

Vassiliev Invariants from Symmetric Spaces

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Abstract

We construct a natural framed weight system on chord diagrams from the curvature tensor of any pseudo-Riemannian symmetric space. These weight systems are of Lie algebra type and realized by the action of the holonomy Lie algebra on a tangent space. Among the Lie algebra weight systems, they are exactly characterized by having the symmetries of the Riemann curvature tensor.

1 Introduction

The essence of this paper is the simple observation that the curvature tensor of a pseudo-Riemannian symmetric space satisfies an identity analogous to the 4T relation in the theory of Vassiliev invariants. This means that the curvature tensor gives rise to a weight system on chord diagrams in the most natural way, and by virtue of the Kontsevich integral, this weight system integrates to a finite-type invariant.

Suppose that (M, g) is a connected pseudo-Riemannian manifold, with Levi-Civita connection ∇ and curvature $R \in C^\infty(M, T^*M \otimes T^*M \otimes \text{End}(TM))$. Using the metric to identify T^*M with TM , we get a tensor $\hat{R} \in C^\infty(M, \text{End}(TM) \otimes \text{End}(TM))$. This can be used to construct a function w_R on chord diagrams, by placing \hat{R} on each chord and contracting around the circle, as done in the following example

$$w_R(\text{chord diagram}) = \text{chord diagram with } \hat{R} \text{ on chords} = \hat{R}_{ae}^{bf} \hat{R}_{cf}^{ad} \hat{R}_{bd}^{ce} \in C^\infty(M).$$

If (M, g) is a symmetric space, the curvature tensor is parallel. This will not only have the obvious implication that w_R produces constant functions on M , but also that it satisfies the 4T relation on chord diagrams. Therefore, it defines a weight system and a finite-type invariant of knots through the universal Vassiliev invariant.

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The algebra of chord diagrams, with the $4T$ relation, can be equivalently represented as closed trivalent Jacobi diagrams, modulo the IHX and AS relations. Through this viewpoint, there is a striking connection to metrized Lie algebras, which can be used to construct weight systems [BN]. This is done by placing the totally antisymmetric structure tensor $Y \in \mathfrak{g}^{\otimes 3}$ at each trivalent internal vertex and contracting edges using the metric. The Jacobi identity ensures that the IHX relation is satisfied, and one obtains a central element in the universal enveloping algebra. By applying a representation of the Lie algebra and taking the trace, this construction produces a numerical weight system. As we shall see, the weight systems defined by symmetric spaces are in fact of this Lie algebra type and realized through the holonomy Lie algebra and its action on the tangent spaces of the symmetric space.

Using the curvature of a hyper-Kähler metric to construct weight systems was previously proposed by Rozansky and Witten [RW]. Soon after, it was realized by Kapranov and Kontsevich [Kap, Kon2] that the construction would also work in the holomorphic symplectic setting, using the Atiyah class in place of the curvature. At each trivalent vertex of a Jacobi diagram, Rozansky and Witten put a copy of the curvature, viewed as one-form $R \in \Omega^{0,1}(M, T^* \otimes T^* \otimes T^*)$, and contracted using the holomorphic symplectic form. The one-form part of R is left out of this contraction, so the result gives a cohomology class of degree equal to the number of trivalent vertices. It can be integrated if the dimension of the manifold is appropriate.

The Bianchi identity plays an important role, analogous to the Jacobi identity, in proving the IHX relation for Rozansky and Witten. In our case, the Bianchi identity plays a central role in proving that the weight systems coming from a symmetric space are of Lie algebra type. It literally ensures the Jacobi identity for the symmetric triple (Definition 4.3) of the symmetric space. On the other hand, it is not needed to see that symmetric spaces give rise to weight systems in the first place. This underlines the curious fact, that weight systems coming from symmetric spaces are most naturally expressed on chord diagrams, whereas the Rozansky-Witten weight systems and Bar-Natan's Lie algebra weights seem to favor the trivalent Jacobi diagrams.

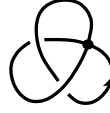
The paper is organized as follows. In Section 2, we recall the basic theory of Vassiliev invariants, establishing notation and introducing the Jones and Yamada polynomials, which will be used to illustrate throughout. The definitive resources on this theory are Bar-Natan's paper [BN] and the excellent book [CDM]. In Section 3, we give a simple framework for constructing weight systems, using what we call a weight tensor, which must satisfy a variant of the $4T$ relation. The ubiquitous weight systems coming from representations of a metrized Lie algebras fit this description, and the weight tensor is obtained by applying the representation to the Casimir tensor. Then we prove the main observation of the paper, which is that the curvature of a pseudo-Riemannian symmetric space defines a weight tensor. Finally, in Section 4, we prove that the weight systems obtained from symmetric spaces are of Lie algebra type and realized through the holonomy Lie algebra. Moreover, we use the correspondence between pseudo-Riemannian symmetric spaces and symmetric Lie algebra triples [CP] to show that a weight tensor of Lie algebra type can be realized on a symmetric space exactly if it carries the symmetries of a curvature tensor.

2 Vassiliev Invariants

A knot invariant is a function on the space \mathcal{K} of ambient isotopy classes of oriented knots in \mathbb{R}^3 . Among such functions, the *Vassiliev* or *finite-type* invariants correspond in a certain sense to polynomial functions. To explicate the polynomial nature of these invariants, one usually considers the larger space of singular knots, which are allowed to have double points.



A knot



A singular knot

The set of singular knots with n singular points is denoted by \mathcal{K}_n . Any knot invariant v can be extended to singular knots through the skein relation

$$v(\text{crossing}) = v(\text{positive crossing}) - v(\text{negative crossing}). \quad (1)$$

A *Vassiliev* invariant of order at most n is a knot invariant whose extension vanishes on any singular knot with at least $n + 1$ singularities. The extension of a knot invariant to singular knots, through the equation (1), can then be viewed as a derivative of the invariant. The restriction to knots with n double points corresponds to the n 'th derivative, so Vassiliev invariants of order n are exactly those with vanishing derivative of order $n + 1$. In this sense they are analogous to polynomials of order n .

The space of Vassiliev invariants of order n is denoted by \mathcal{V}_n . Clearly we have a natural filtration,

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \dots \subset \bigcup_{n \geq 0} \mathcal{V}_n = \mathcal{V}.$$

Example 2.1. The Jones polynomial is an invariant of oriented links, with values in the Laurent polynomials $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$. It can be defined by the skein relations

$$t^{-1}J(\text{crossing}) - tJ(\text{crossing}) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})J(\text{two strands}) \quad \text{and} \quad J(\text{circle}) = 1.$$

It follows easily that the Jones polynomial of a knot actually takes values in $\mathbb{Z}[t, t^{-1}]$. After substituting $t = e^h$ and expanding the Jones polynomial as a power series in h , the coefficient j_n of h^n turns out to be a Vassiliev invariant of order n [BL, BN]. Indeed, it is easy to check that

$$J(\text{crossing}) = J(\text{positive crossing}) - J(\text{negative crossing}) = h(\dots),$$

so the Jones polynomial of a knot K with $n + 1$ singularities will be divisible by h^{n+1} , and in particular $j_n(K)$ will vanish. The invariants j_n are called the Jones invariants. \diamond

The Jones polynomial is of course not a Vassiliev itself, but we can view it as a formal power series with coefficients in Vassiliev invariants. To capture such invariants, one introduces the space of *power series* Vassiliev invariants

$$\widehat{\mathcal{V}} := \prod_{n \geq 0} \mathcal{V}_n.$$

The natural pointwise multiplication of invariants turns the space \mathcal{V} of all Vassiliev invariants into a filtered algebra. In fact, this space also has a natural coproduct $\Delta: \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}$, dual to the operation of connect sum of knots,

$$\Delta(v)(K_1, K_2) = v(K_1 \# K_2).$$

This gives \mathcal{V} the structure a filtered bialgebra.

To any singular knot, we can assign a chord diagram on the circle, encoding the order of singular points along the knot. With A_n being the set of chord diagrams with n chords, this assignment defines a map

$$\delta: \mathcal{K}_n \rightarrow A_n \quad \text{e.g.} \quad \begin{array}{c} \text{[Knot Diagram]} \end{array} \mapsto \begin{array}{c} \text{[Chord Diagram]} \end{array}$$

The circle represents the knot, and chords connect points corresponding to the same double point. If $K \in \mathcal{K}_n$ is a knot with n singularities and $v \in \mathcal{V}_n$ is a Vassiliev invariant of order at most n , then $v(K)$ is invariant under crossing changes on K . In other words, the evaluation $v(K)$ only depends on the order of the singularities along the knot, or simply its associated chord diagram. This means that v induces a map, called the *symbol* of v ,

$$\sigma_n(v): A_n \rightarrow \mathbb{C}, \quad \sigma_n(v)(D) = v(K_D),$$

where $K_D \in \mathcal{K}_n$ is any knot with $\delta(K_D) = D$.

Not every map on chord diagrams can be realized as the symbol of a Vassiliev invariant. Indeed, it is not difficult to show that such a symbol must satisfy the conditions of the following definition.

Definition 2.2. Any map $w: A_n \rightarrow \mathbb{C}$ satisfying the 4T relation,

$$w(\text{[Diagram 1]}) - w(\text{[Diagram 2]}) + w(\text{[Diagram 3]}) - w(\text{[Diagram 4]}) = 0, \quad (2)$$

is called a (framed) *weight system* of order n . If in addition it satisfies the 1T relation,

$$w(\text{[Diagram 5]}) = 0, \quad (3)$$

then it is called an *unframed* weight system.

The diagrams in the 4T relation can have other chords with endpoints on the dotted parts of the circle and possibly intersecting the shown chords. The same holds for the 1T relation, except the other chords are not allowed to intersect the shown chord, meaning they must stay in the gray regions.

The vector space of (unframed) weight systems of order n is denoted by \mathcal{W}_n . Since the symbol of a Vassiliev invariant defines a weight system, we see that the symbol defines a map $\sigma_n: \mathcal{V}_n \rightarrow \mathcal{W}_n$. Clearly two Vassiliev invariants of order n have the same symbol if and only if their difference is in \mathcal{V}_{n-1} , which is therefore the kernel of σ_n . This means that the symbol descends to an injective map

$$\bar{\sigma}_n: \mathcal{V}_n / \mathcal{V}_{n-1} \rightarrow \mathcal{W}_n,$$

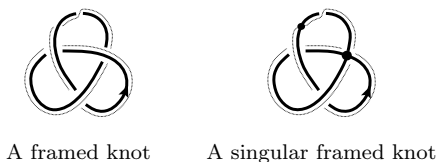
which is in fact an isomorphism. Indeed, surjectivity follows immediately from the following fundamental theorem of Kontsevich [Kon1].

Theorem 2.3. *There exists a map $J_n: \mathcal{W}_n \rightarrow \mathcal{V}_n$ such that $\sigma_n \circ J_n = \text{Id}$.*

The proof uses the celebrated Kontsevich integral to construct the invariants on Morse knots. There is also a combinatorial construction using the Drinfeld associator [Car, Piu1]. The resulting map J_n yields a preferred Vassiliev invariant having a given weight system as its symbol, and such invariants in the image of J_n are called *canonical*. In fact, the Jones invariants described in Example 2.1 are canonical. This follows by their relation to quantum invariants, to which we shall return, and the work of Le-Murakami and Kassel [LM, Kas]. The theorem above is often formulated in terms of the so-called universal Vassiliev invariant. We shall review this description after briefly recalling the situation for framed knots.

Framed Knots

The theory of Vassiliev invariants also applies to framed knots, and in some sense more naturally. In this case, the framing for a singular knot is allowed to have simple zeros away from the double points.



The set of singular knots with n singular points, of the knot or the framing, is denoted by \mathcal{K}_n^φ . Framed knot invariants are extended to singular knots by resolving singularities of the framing through the relation

$$v\left(\begin{array}{c} \text{ } \\ \text{ } \end{array}\right) = v\left(\begin{array}{c} \text{ } \\ \text{ } \end{array}\right) - v\left(\begin{array}{c} \text{ } \\ \text{ } \end{array}\right). \quad (4)$$

A framed Vassiliev invariant of order n vanishes on all knots with at least $n + 1$ singularities. The space of such invariants is denoted by \mathcal{V}_n^φ .

Example 2.4. The Yamada polynomial [Yam] is an invariant of framed unoriented links, with values in $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. It is defined by the skein relations

$$Y(\text{diagram 1}) - Y(\text{diagram 2}) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \left(Y(\text{diagram 3}) - Y(\text{diagram 4}) \right)$$

with framing and initial conditions

$$q^{-1} Y(\text{diagram 5}) = q Y(\text{diagram 6}) = Y(\text{diagram 7}) \quad \text{and} \quad Y(\text{diagram 8}) = q^{\frac{1}{2}} + q^{-\frac{1}{2}} + 1.$$

This is a specialization of the two-variable Dubrovnik polynomial, which is a variant of the Kauffman polynomial. Of course the Yamada polynomial defines an invariant of oriented knots simply by ignoring the orientation. In fact, the Yamada polynomial is essentially given by evaluating the Jones polynomial on the $(2,0)$ -cabling of a knot [Yam].

Performing the substitution $q = e^h$ and expanding in powers of h , the coefficient y_n of h^n will be a Vassiliev invariant of order n . Indeed, one checks that

$$Y(\text{diagram 9}) = Y(\text{diagram 10}) - Y(\text{diagram 11}) = h(\dots),$$

so the Yamada polynomial of a knot K with $n+1$ double points will be divisible by h^{n+1} , and in particular $y_n(K)$ will vanish. \diamond

When calculating the value of a Vassiliev invariant $v \in \mathcal{V}_n^\varphi$ on a framed knot $K \in \mathcal{K}_n^\varphi$, the zeros of the framing can be replaced by double points of the knot, simply because

$$v(\text{diagram 12}) = \frac{1}{2} v(\text{diagram 13}) - \frac{1}{2} v(\text{diagram 14}) = \frac{1}{2} v(\text{diagram 15}).$$

This means that we can use the same chord diagrams in the framed case. The map $\delta^\varphi: \mathcal{K}_n^\varphi \rightarrow A_n$ sends a zero of the framing to a chord connecting two adjacent points on the circle. As in the unframed case, we therefore get a symbol map

$$\sigma_n^\varphi(v): A_n \rightarrow \mathbb{C}, \quad \sigma_n^\varphi(v)(D) = v(K_D),$$

where $K_D \in \mathcal{K}_n^\varphi$ is any framed knot with $\delta^\varphi(K_D) = D$.

Example 2.5. Let us calculate the symbol of the Yamada polynomial. First of all, we observe that

$$Y(\text{diagram 16}) = Y(\text{diagram 17}) - Y(\text{diagram 18}) = h \left(y_0(\text{diagram 19}) - y_0(\text{diagram 20}) \right) + O(h^2).$$

This means that y_n evaluated on a link with n double points is given by the signed sum of y_0 evaluated on all possible ways of smoothing the double points vertically or horizontally. To write down a formula, let s denote a map from the double points of a given link L to the set $\{1, -1\}$, and let L_s be the link obtained by smoothing each double point of L according to

the rule

$$\begin{array}{c} \text{---} \circlearrowleft \text{---} \\ \text{---} \circlearrowright \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \text{if } s(d) = 1 \quad \text{and} \quad \begin{array}{c} \text{---} \circlearrowleft \text{---} \\ \text{---} \circlearrowright \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \text{if } s(d) = -1.$$

If $(-1)^{|s|}$ denotes the sign of s , defined as the product of its values on all double points, then we have

$$y_n(L) = \sum_s (-1)^{|s|} y_0(L_s) = \sum_s (-1)^{|s|} 3^{c(L_s)},$$

where the sum is over all maps s , and $c(L_s)$ denotes the number of components of the smoothing L_s . The fact that $y_0(L) = c(L)$ for any link L is easily verified.

On the level of chord diagrams, this can be visualized in the following way. Once again, let s denote a map from the chords of a given diagram D to the set $\{1, -1\}$, and let D_s be the collection of circles obtained by smoothing each chord according to the rule

$$\begin{array}{c} \text{---} \circlearrowleft \text{---} \\ \text{---} \circlearrowright \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \text{if } s(c) = 1 \quad \text{and} \quad \begin{array}{c} \text{---} \circlearrowleft \text{---} \\ \text{---} \circlearrowright \text{---} \end{array} \rightsquigarrow \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \text{if } s(c) = -1.$$

Then the symbol of y_n is given by

$$\sigma_n^\varphi(y_n)(D) = \sum_s (-1)^{|s|} 3^{c(D_s)},$$

where the sum is over all maps s on the chords of D , and $c(D_s)$ is the number of components of the resolved diagram D_s . \diamond

The symbol of a framed invariant satisfies the 4T relation, but not the 1T relation. Indeed, if \mathcal{W}_n^φ denotes the set of (framed) weight systems, then the symbol descends to a map

$$\bar{\sigma}_n^\varphi: \mathcal{V}_n^\varphi / \mathcal{V}_{n-1}^\varphi \rightarrow \mathcal{W}_n^\varphi, \tag{5}$$

and Theorem 2.3 has the following analogue.

Theorem 2.6. *There exists a map $J_n^\varphi: \mathcal{W}_n^\varphi \rightarrow \mathcal{V}_n^\varphi$ such that $\sigma_n^\varphi \circ J_n^\varphi = \text{Id}$.*

The first proof of this is due to Le and Murakami [LM], and employs a combinatorial description using the Drinfeld associator. An alternative proof, using a framed version of the Kontsevich integral, is provided by Goryunov [Gor]. Once again, invariants in the image of J^φ are called canonical.

Through the work of Reshetikhin and Turaev [Tur, RT], representations of quantum groups provide a rich source of framed knot invariants. This elaborate construction associates an invariant of framed knots to any representation of a semi-simple Lie algebra \mathfrak{g} . In fact, the representation of \mathfrak{g} can be deformed to a representation of the quantum group $U_q(\mathfrak{g})$, yielding a solution to the Yang-Baxter equation which is then used in constructing the invariant.

Example 2.7. For the standard representation of \mathfrak{sl}_2 , the resulting quantum invariant $Q_{\mathfrak{sl}_2}^\varphi$ can be characterized by the skein relation

$$q^{\frac{1}{4}} Q_{\mathfrak{sl}_2}^\varphi(\text{crossing}) - q^{-\frac{1}{4}} Q_{\mathfrak{sl}_2}^\varphi(\text{crossing}) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) Q_{\mathfrak{sl}_2}^\varphi(\text{two parallel strands})$$

with the framing and initial conditions

$$Q_{\mathfrak{sl}_2}^\varphi(\text{loop}) = q^{\frac{3}{2}} Q_{\mathfrak{sl}_2}^\varphi(\text{strand}) \quad \text{and} \quad Q_{\mathfrak{sl}_2}^\varphi(\text{circle}) = q^{\frac{1}{2}} + q^{-\frac{1}{2}}.$$

There is a standard way of deframing general quantum invariants to produce invariants of unframed knots. For $Q_{\mathfrak{sl}_2}^\varphi$ this deframing procedure amounts to

$$Q_{\mathfrak{sl}_2}(K) = q^{-3/4 \cdot w(K)} Q_{\mathfrak{sl}_2}^\varphi(K),$$

where K is a knot diagram and $w(K)$ is its writhe, or simply the difference between the number of positive and negative crossings in the diagram. This means that the deframed invariant satisfies

$$q Q_{\mathfrak{sl}_2}(\text{crossing}) - q^{-1} Q_{\mathfrak{sl}_2}(\text{crossing}) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) Q_{\mathfrak{sl}_2}(\text{two parallel strands}),$$

so in fact the Jones polynomial of a knot K can be expressed as

$$J(K) = (t^{\frac{1}{2}} + t^{-\frac{1}{2}})^{-1} Q_{\mathfrak{sl}_2}(K)|_{q=t^{-1}}.$$

In this sense, the quantum invariant $Q_{\mathfrak{sl}_2}^\varphi$ constitutes a natural framed version of the Jones polynomial. \diamond

Example 2.8. The quantum invariant $Q_{\mathfrak{so}_3}^\varphi$ arising from the standard representation of \mathfrak{so}_3 is equal to the Yamada polynomial described in Example 2.4 (see [BB]). \diamond

It is a general result of Birman and Lin [BL] that any quantum invariant produces a power series Vassiliev invariant by substituting $q = e^h$ and expanding in h . We saw examples of this in Example 2.1 and Example 2.4. Moreover, the work of Le-Murakami and Kassel [LM, Kas] establishes that these finite-type invariants are in fact canonical. In particular, this holds for the Yamada polynomial and for the Jones polynomial, as we have already mentioned.

The Universal Vassiliev Invariant

The universal Vassiliev invariant gives a unified way of encoding the maps J_n^φ of Theorem 2.6. We shall focus on the framed case, but the description works equally well in the unframed setting. To describe the universal Vassiliev invariant, it is convenient to consider the 4T relation on the vector space spanned by chord diagrams.

Definition 2.9. The space of (framed) chord diagrams \mathcal{A}_n^φ of order n is the vector space $\mathbb{C}A_n$, spanned by diagrams in A_n , modulo the subspace spanned by the 4T relations

$$\begin{array}{c} \text{Diagram 1} \end{array} - \begin{array}{c} \text{Diagram 2} \end{array} + \begin{array}{c} \text{Diagram 3} \end{array} - \begin{array}{c} \text{Diagram 4} \end{array} = 0. \quad (6)$$

The space of unframed chord diagrams is the vector space $\mathcal{A} = \mathcal{A}^\varphi / (\Theta)$, where (Θ) is the subspace spanned by the 1T relations,

$$\begin{array}{c} \text{Diagram 5} \end{array} = 0. \quad (7)$$

The space $\mathcal{A}^\varphi = \bigoplus_n \mathcal{A}_n^\varphi$ has a natural graded multiplication given by connected sum of chord diagrams, which is only well-defined up to the 4T relation. With this multiplication, the space (Θ) above is exactly the ideal generated by the chord diagram Θ with a single chord. In addition, the algebra \mathcal{A}^φ has a graded coproduct given by

$$\Delta(D) = \sum_{J \subseteq \{D\}} D_J \otimes D_{\bar{J}}, \quad (8)$$

where the sum is over all subsets J of chords in D , and D_J only has chords of J , whereas $D_{\bar{J}}$ only has the complementary set of chords. This gives \mathcal{A}^φ the structure of a Hopf algebra.

The dual of \mathcal{A}_n^φ is clearly just the space of weight systems $\mathcal{W}_n^\varphi = \text{Hom}(\mathcal{A}_n^\varphi, \mathbb{C})$, and through this duality the Hopf algebra structure on \mathcal{A}^φ induces a dual structure of the above type on $\mathcal{W}^\varphi = \bigoplus_n \mathcal{W}_n^\varphi$. The isomorphisms in (5) respect the algebraic structure in the sense that \mathcal{W}^φ is isomorphic to the associated graded of the filtered bialgebra \mathcal{V}^φ ,

$$\bar{\sigma}^\varphi : \bigoplus \mathcal{V}_n^\varphi / \mathcal{V}_{n-1}^\varphi \xrightarrow{\sim} \bigoplus \mathcal{W}_n^\varphi.$$

Through the duality between \mathcal{A}_n^φ and \mathcal{W}_n^φ , the map $J_n^\varphi : \mathcal{W}_n^\varphi \rightarrow \mathcal{V}_n^\varphi$ of Theorem 2.6 is equivalent to a map

$$I_n^\varphi : \mathcal{K}^\varphi \rightarrow \mathcal{A}_n^\varphi, \quad J_n^\varphi(w)(K) = w(I_n^\varphi(K)).$$

Actually, the proof of Theorem 2.6 constructs this map I_n^φ rather than J_n^φ . Since J_n^φ takes values in \mathcal{V}_n^φ , the map I_n^φ is a Vassiliev invariant of order n , with values in \mathcal{A}_n^φ . The maps I_n^φ are usually collected in a single map, called the *universal Vassiliev invariant*,

$$I^\varphi : \mathcal{K}^\varphi \rightarrow \hat{\mathcal{A}}^\varphi,$$

where $\hat{\mathcal{A}}^\varphi = \prod_n \mathcal{A}_n^\varphi$ is the graded completion of the algebra \mathcal{A}^φ of chord diagrams. Appropriately normalized, this map is actually multiplicative with respect to the connected sum of knots (see [CDM]). Moreover, the coefficients of I^φ are in fact rational [LM].

By a standard symbol calculus argument, it is easy to show that any Vassiliev invariant factors through the universal one.

Proposition 2.10. *For any $v \in \mathcal{V}^\varphi$, there is a unique $w \in \mathcal{W}^\varphi$ such that $v = w \circ I^\varphi$.*

In this sense, the universal Vassiliev invariant allows us to speak about the lower order symbols of any Vassiliev invariant in \mathcal{V}^φ . Moreover, applying the maps J_n^φ to this total symbol, any Vassiliev invariant can be uniquely written as

$$v = v_n^c + \cdots + v_1^c + v_0^c,$$

where $v_i^c \in \mathcal{V}_i^\varphi$ are canonical Vassiliev invariants. In fact, this establishes a bijective correspondence between the subspace of $\widehat{\mathcal{V}}^\varphi$ consisting of canonical power series Vassiliev invariants and weight systems in the graded completion $\widehat{\mathcal{W}}^\varphi = \prod_n \mathcal{W}_n^\varphi$.

3 Constructing Weight Systems

In this section, we shall use the geometry of pseudo-Riemannian symmetric spaces to construct framed weight systems on chord diagrams. Through the Kontsevich integral, these will in turn give rise to Vassiliev invariants. As we shall see, the construction does not yield new weight systems, but rather weights of the familiar Lie algebra type.

We start by describing a simple general recipe for constructing weight systems. Suppose that V is a finite-dimensional vector space and consider a symmetric tensor,

$$H \in \text{End}(V) \otimes \text{End}(V).$$

Symmetry means that H is invariant under the involution that interchanges the two copies of the endomorphism space, or simply $H_{ac}^{bd} = H_{ca}^{db}$ in index notation. Such a tensor can be used to construct a function $w_H: A \rightarrow \mathbb{C}$ on chord diagrams. The value on a particular chord diagram $D \in A_n$ is given by a full contraction of n copies of H as prescribed by the chords of D . This can be explained using diagram by representing the tensor H in the following graphical way

$$H_{ac}^{bd} = \begin{array}{c} \text{c} \\ \curvearrowright \text{H} \curvearrowleft \\ \text{d} \quad \text{a} \end{array} \quad \text{or simply} \quad H = \begin{array}{c} \curvearrowleft \text{H} \curvearrowright \end{array}. \quad (9)$$

Given a chord diagram, we view it as being glued from such pieces, and contract the tensors accordingly. For example, we have

$$w_H(\text{chord diagram}) = \begin{array}{c} \text{b} \\ \text{c} \quad \text{a} \\ \text{d} \quad \text{f} \\ \text{e} \end{array} = H_{ae}^{bf} H_{cf}^{ad} H_{bd}^{ce},$$

where repeated indices denote contraction of tensor entries, or simply Einstein summation if we think of the indices as indexing a basis of the vector space V . The symmetry of the tensor H ensures that the diagrammatic representation is well-defined.

Using the following graphical representation, the map w_H is even more naturally expressed on singular knots, where the tensor H can be placed at every double point and contracted as prescribed by the knot,

$$H = \begin{array}{c} \uparrow \uparrow \\ \boxed{H} \\ \downarrow \downarrow \end{array} \quad \text{and} \quad w_H \left(\text{diagram of a knot with two crossings} \right) = \text{diagram of the same knot with } H \text{ boxes at the crossings}. \quad (10)$$

For a general tensor H , the function w_H will not necessarily satisfy the 4T relation and define a weight system.

Definition 3.1. A *weight tensor* is a symmetric element $H \in \text{End}(V) \otimes \text{End}(V)$ which satisfies the 4T relation

$$H_{ea}^{fx} H_{xc}^{bd} - H_{ex}^{fb} H_{ac}^{xd} + H_{ec}^{fx} H_{ax}^{bd} - H_{ex}^{fd} H_{ac}^{bx} = 0 \quad (11)$$

as an element in $\text{End}(V) \otimes \text{End}(V) \otimes \text{End}(V)$.

In terms of the graphical representation in (9), the 4T relation corresponds to

$$\text{diagram 1} - \text{diagram 2} + \text{diagram 3} - \text{diagram 4} = 0. \quad (12)$$

The point of this definition is, of course, the following obvious statement:

Proposition 3.2. Any weight tensor H defines a framed weight system $w_H: \mathcal{A}^\varphi \rightarrow \mathbb{C}$.

Weight Systems from Metrized Lie Algebras

There is a standard way of producing a framed weight system from a representation of a metrized Lie algebra. The construction is due to Bar-Natan [BN] and proceeds as follows.

Let \mathfrak{g} be a Lie algebra equipped with a non-degenerate bilinear form $B \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ which is invariant under the adjoint action, i.e.,

$$B([z, x], y) + B(x, [z, y]) = 0,$$

for all $x, y, z \in \mathfrak{g}$. Since B is non-degenerate, its inverse constitutes an element $C \in \mathfrak{g} \otimes \mathfrak{g}$, called the Casimir tensor. If

$$\rho: \mathfrak{g} \rightarrow \text{End}(V)$$

is a representation of \mathfrak{g} , then its tensor square can be applied to C to produce a tensor

$$\rho(C) \in \text{End}(V) \otimes \text{End}(V).$$

This is a weight tensor. Indeed, if $Y \in \mathfrak{g}^{\otimes 3}$ denotes the totally anti-symmetric structure tensor of \mathfrak{g} , given by raising both indices on the Lie bracket using the metric, then $\rho(C)$ satisfies the 4T relation because

$$\rho(C)_{ae}^{xf} \rho(C)_{xc}^{bd} - \rho(C)_{ac}^{xd} \rho(C)_{xe}^{bf} = \rho(Y)_{ace}^{bdf} = \rho(C)_{ac}^{bx} \rho(C)_{xe}^{df} - \rho(C)_{ax}^{bd} \rho(C)_{ce}^{xf}. \quad (13)$$

Graphically, this corresponds to

$$\begin{array}{c} \text{Diagram 1} \end{array} - \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array} = \begin{array}{c} \text{Diagram 4} \end{array} - \begin{array}{c} \text{Diagram 5} \end{array} \quad (14)$$

where the trivalent vertex in the middle represents the trivector $Y \in \mathfrak{g}^{\otimes 3}$. From Proposition 3.2, we immediately get the following:

Proposition 3.3. *Any representation $\rho: \mathfrak{g} \rightarrow \text{End}(V)$ of a metrized Lie algebra defines a framed weight system $w_\rho: \mathcal{A}^\varphi \rightarrow \mathbb{C}$.*

The weight systems obtained in this way are said to be of Lie algebra type. A more complicated way of obtaining a weight system from a Lie algebra representation is through the corresponding quantum invariant. By the result of Birman and Lin [BL], the quantum invariant defined from a representation of a Lie algebra \mathfrak{g} gives rise to a power series Vassiliev invariant through the substitution $q = e^h$. Conveniently, the weight system of this power series invariant matches the weight system constructed directly from the Lie algebra representation [Piu2].

Example 3.4. The Lie algebra \mathfrak{sl}_2 has the standard generators

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and the Casimir tensor of the non-degenerate form $B(x, y) = \text{Tr}(xy)$ is given by

$$C = \frac{1}{2}H \otimes H + E \otimes F + F \otimes E. \quad (15)$$

The weight tensor associated with the standard representation is of course given by the same expression, and it agrees with the symbol of the power series Vassiliev invariant coming from the quantum invariant $Q_{\mathfrak{sl}_2}^\varphi$, as discussed in Example 2.7. A combinatorial description of this weight system, along the lines of Example 2.5, is given in [CDM]. \diamond

Example 3.5. The weight system arising from the standard representation of the Lie algebra \mathfrak{so}_3 , with non-degenerate and invariant bilinear form $B(x, y) = \frac{1}{2} \text{Tr}(xy)$, is equal to the weight system of the Yamada polynomial, described in Example 2.4 and Example 2.5 (see [BN, CDM]). This is in agreement with Birman and Lin [BL] and the fact that the Yamada polynomial corresponds to the quantum invariant coming from \mathfrak{so}_3 with its standard representation [BB]. \diamond

Weight Systems from Symmetric Spaces

We shall see that pseudo-Riemannian symmetric spaces give rise to a framed weight system in the simplest way possible: the curvature defines a weight tensor.

Let (M, g) denote a pseudo-Riemannian manifold, with Levi-Civita connection ∇ and curvature R , given as usual by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

for any vector fields X, Y, Z on M .

Definition 3.6. A (locally) *symmetric space* is a connected pseudo-Riemannian manifold (M, g) with parallel curvature tensor, that is $\nabla R = 0$.

The justification of this name lies in the standard fact that the curvature is parallel if and only if the geodesic reflections are local isometries. A special case is when these reflections extend to global isometries of the manifold.

Definition 3.7. A *globally symmetric space* is a pseudo-Riemannian manifold (M, g) such that every point $p \in M$ is an isolated fixed point of an involutive isometry s_p .

In addition to the obvious antisymmetry $R_{abc}^d = -R_{bac}^d$, the curvature tensor satisfies the algebraic Bianchi identity

$$R_{abc}^d + R_{bca}^d + R_{cab}^d = 0,$$

as well as the differential Bianchi identity on the covariant derivative of R , which is vacuous in the case of symmetric spaces since the curvature is parallel.

To expose the remaining symmetries of the curvature, it is common to lower the index and consider the fully covariant curvature tensor

$$\check{R}_{abcd} = R_{abc}^x g_{xd} \quad \text{which satisfies} \quad \check{R}_{abcd} = \check{R}_{cdab}.$$

We shall, however, be interested in the tensor $\hat{R} \in C^\infty(M, \text{End}(TM) \otimes \text{End}(TM))$ given by raising an index of R ,

$$\hat{R}_{ac}^{bd} = g^{bx} R_{axc}^d.$$

The symmetry of \check{R} translates to the symmetry $\hat{R}_{ac}^{bd} = \hat{R}_{ca}^{db}$, so \hat{R} defines a symmetric section of $\text{End}(TM) \otimes \text{End}(TM)$. Moreover, we have the following simple but crucial result, which is the main observation of the paper.

Proposition 3.8. *The curvature \hat{R} of a locally symmetric space is a weight tensor.*

Proof. We must verify the 4T relation (11). Since the curvature tensor is parallel, we have

$$\begin{aligned} 0 &= \nabla_f \nabla_e R_{abc}^d - \nabla_e \nabla_f R_{abc}^d \\ &= R_{efa}^x R_{xbc}^d + R_{efb}^x R_{axc}^d + R_{efc}^x R_{abx}^d - R_{efx}^d R_{abc}^x. \end{aligned} \tag{16}$$

Raising the indices b and f with the metric, we obtain

$$0 = \hat{R}_{ea}^{fx} \hat{R}_{xc}^{bd} - \hat{R}_{ex}^{fb} \hat{R}_{ac}^{xd} + \hat{R}_{ec}^{fx} \hat{R}_{ax}^{bd} - \hat{R}_{ex}^{fd} \hat{R}_{ac}^{bx},$$

which is the desired identity. \square

The main theorem follows as an immediate corollary of the above proposition.

Theorem 3.9. *The curvature \hat{R} of a locally symmetric space defines a framed weight system $w_R: \mathcal{A}^\varphi \rightarrow \mathbb{R}$.*

Proof. By Proposition 3.8, the curvature tensor \hat{R} satisfies the 4T relation on the tangent space at every point on the symmetric space M . This means that \hat{R} defines a map $w_R: \mathcal{A}^\varphi \rightarrow C^\infty(M)$. For a given chord diagram $D \in \mathcal{A}_n^\varphi$, the function $w_R(D)$ is given by a full contraction of the curvature and the metric, both of which are parallel. This means that the derivative of $w_R(D)$ must vanish so that w_R takes values in constant functions. \square

Through these weight systems and the universal Vassiliev invariant, any symmetric space gives rise to a power series Vassiliev invariant.

Remark 3.10. In fact, the proof of Proposition 3.8 does not require the curvature to be parallel, but only that its covariant derivative $\nabla R \in \Omega^1(M, (TM^*)^{\otimes 3} \otimes TM)$ defines a closed one-form with respect to the covariant exterior derivative d^∇ . It would be interesting to find examples of non-symmetric spaces of this type. The weight system coming from such a space would in general take values in functions on M , which could be integrated to produce numbers for compact spaces.

In general, the weight systems coming from symmetric spaces will detect the framing. Indeed, the value on the chord diagram with a single chord will be given by the scalar curvature, so the weight system will be framed unless the symmetric space is scalar flat. Since any irreducible symmetric space is Einstein ([Pet], Theorem 57), it is only scalar flat if it is Ricci flat, in which case it is flat altogether ([Bes], Theorem 7.61).

Example 3.11. The three-sphere $S^3 = \{x \in \mathbb{R}^4 \mid |x| = 1\}$, with the subspace metric g induced from the standard Euclidean metric on \mathbb{R}^4 , is a simply connected Riemannian symmetric space. The sectional curvature is constant and equal to 1, so the curvature tensor is given by

$$\hat{R}_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd} \quad \text{or equivalently} \quad \hat{R}_{ac}^{bd} = \mathbb{1}_a^d \mathbb{1}_c^b - g_{ac}g^{bd}. \quad (17)$$

This is easily seen to imply that the corresponding weight system is given by

$$w_R(D) = \sum (-1)^{|s|} 3^{c(D_s)},$$

which agrees with that of the Yamada polynomial described in Example 2.5. Indeed, the particular form of the curvature (17) means that evaluating the corresponding weight system on a singular knot, as in (10), yields the signed sum over all ways of smoothing the double point horizontally and vertically and taking the trace of the identity endomorphism, which is equal to 3, for each the resulting components. \diamond

It is no coincidence that the weight system for S^3 is of Lie algebra type. As we shall see, this is in fact the case for any weight system constructed from the curvature tensor of a symmetric space.

4 Symmetric Spaces and Lie Algebras

Representations of Lie algebras enter the picture in a natural way for globally symmetric spaces, which are homogeneous spaces for the action of the isometry group. An isotropy

subgroup has an infinitesimal action of its Lie algebra on the tangent space at the fixed point. Now, there are many ways of representing a homogeneous space as a quotient $M = G/H$, and the connection between symmetric spaces and Lie algebra weight systems is the fact that for such a representation the curvature of the symmetric space can be calculated from the bracket on the Lie algebra \mathfrak{g} of G as

$$R(X, Y)Z = -[[X, Y], Z] \quad \forall X, Y, Z \in T_p M \subset \mathfrak{h} \oplus T_p M = \mathfrak{g}.$$

In general, however, the Lie algebra of G is not metrized in any natural way, and we need a metrized Lie algebra to produce weight systems. It turns out that a more careful representation of the symmetric space as homogeneous for the transvection group resolves this issue. For positive-definite Riemannian symmetric spaces, the transvection group coincides with the connected component of the isometry group, but not in general. In this representation, the stabilizer of a point is isomorphic to the holonomy group and its Lie algebra carries a natural metric. The Lie algebra of the holonomy group also has the advantage that it makes sense for locally symmetric spaces and does not rely on the homogeneous description.

As we have seen, the weight system constructed from a symmetric space is determined by the metric and the curvature on a single tangent space, and does not depend on the global geometry of the symmetric space. The Lie algebras of the transvection and holonomy groups can be described directly from this linear data, avoiding the use of Lie group theory for symmetric spaces. For the general theory of symmetric spaces, the reader is referred to [Hel, Pet, Bal] and in particular to [CP, Neu, KO] for the specific theory used below.

Let (M, g) be any locally symmetric space with curvature R . Fix a base point $p \in M$, and denote the tangent space at p by \mathfrak{p} . Let \mathfrak{h} be the subspace of $\text{End}(\mathfrak{p})$ defined by

$$\mathfrak{h} = \text{span}\{R(X, Y) \in \text{End}(\mathfrak{p}) \mid X, Y \in \mathfrak{p}\}. \quad (18)$$

By the Ambrose-Singer theorem, \mathfrak{h} is in fact the Lie algebra of the holonomy group at p . The Lie bracket is given by the usual commutator of endomorphisms, which can be seen to preserve \mathfrak{h} by rewriting (16) as

$$[R(X, Y), R(Z, W)] = R(R(X, Y)Z, W) + R(Z, R(X, Y)W) \quad (19)$$

for $X, Y, Z, W \in \mathfrak{p}$. This can be viewed as a variant of the 4T relation.

Now, consider the sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and extend the Lie bracket on \mathfrak{h} by

$$\begin{aligned} [X, Y] &= R(X, Y) & \text{for } X, Y \in \mathfrak{p} \\ [A, X] &= -[X, A] = A(X) & \text{for } A \in \mathfrak{h}, X \in \mathfrak{p}. \end{aligned}$$

This is clearly skew-symmetric. Moreover, it satisfies the Jacobi identity, which reduces to (19) for the case of $X, Y \in \mathfrak{p}$ and $A \in \mathfrak{h}$, and to the Bianchi identity of R for the case of $X, Y, Z \in \mathfrak{p}$. In other words, the bracket defines a Lie algebra structure on \mathfrak{g} , and we have

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] = \mathfrak{h}.$$

Notice in particular that \mathfrak{h} equals $[\mathfrak{p}, \mathfrak{p}]$ by definition, and that we get a representation of \mathfrak{h} on \mathfrak{p} by the adjoint action.

Using the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, we define an involution

$$s: \mathfrak{g} \rightarrow \mathfrak{g} \quad \text{by} \quad s(A + X) = A - X \quad \text{for} \quad A \in \mathfrak{h}, X \in \mathfrak{p}.$$

The metric on M defines a non-degenerate bilinear form $B_{\mathfrak{p}}$ on \mathfrak{p} , and this pairing is invariant under the adjoint action of \mathfrak{h} , by the symmetries of the curvature. Moreover, there is a unique extension of $B_{\mathfrak{p}}$ to \mathfrak{g} which is invariant under the involution s and the adjoint action of \mathfrak{g} on itself [CP]. By invariance under s , the summands \mathfrak{h} and \mathfrak{p} must be orthogonal, and on $\mathfrak{h} = [\mathfrak{p}, \mathfrak{p}]$ the invariance under the adjoint action forces

$$B_{\mathfrak{h}}([X, Y], [Z, W]) = B_{\mathfrak{p}}([X, Y], [Z, W]) = \check{R}(X, Y, Z, W). \quad (20)$$

The extension $B = B_{\mathfrak{h}} + B_{\mathfrak{p}}$ is actually non-degenerate. Indeed, if $A \in \mathfrak{h} \subset \text{End}(\mathfrak{p})$ and

$$0 = B_{\mathfrak{h}}(A, [X, Y]) = B_{\mathfrak{p}}([A, X], Y) = B_{\mathfrak{p}}(A(X), Y) \quad \forall X, Y \in \mathfrak{p},$$

then A must vanish by the non-degeneracy of $B_{\mathfrak{p}}$.

In conclusion, we have a metrized Lie algebra \mathfrak{h} and a representation $\rho: \mathfrak{h} \rightarrow \text{End}(\mathfrak{p})$ given by the adjoint action on \mathfrak{g} . The inverse of the non-degenerate metric $B_{\mathfrak{h}}$ defines an element $C_{\mathfrak{h}} \in \mathfrak{h} \otimes \mathfrak{h}$, and (20) has the reformulation

$$\rho(C_{\mathfrak{h}}) = \hat{R},$$

as tensors in $\text{End}(\mathfrak{p}) \otimes \text{End}(\mathfrak{p})$. This proves the following:

Theorem 4.1. *Weight systems from symmetric spaces are of Lie algebra type.*

Theorem 4.1 explains the observation in Example 3.11 above:

Example 4.2. The round unit sphere S^3 has isometry group $O(4)$ with isotropy group $O(3)$. The transvection group is the connected component containing the identity, namely $SO(4)$, and its isotropy subgroup $SO(3)$ gives the holonomy. Therefore, the holonomy Lie algebra is \mathfrak{so}_3 , which explains the equality of its weight system with that of the Yamada polynomial. \diamond

Realizing Weight Systems on Symmetric Spaces

To understand which Lie algebra weight systems can be realized on symmetric spaces, we shall introduce the following terminology [CP].

Definition 4.3. A *symmetric triple* (\mathfrak{g}, s, B) consists of

- a finite-dimensional real Lie algebra \mathfrak{g} ,
- an involutive automorphism $s: \mathfrak{g} \rightarrow \mathfrak{g}$, and
- a non-degenerate, symmetric bilinear form B , which is invariant in the sense that

$$B([x, y], z) = B(x, [y, z]) \quad \text{and} \quad B(sx, sy) = B(x, y) \quad \forall x, y, z \in \mathfrak{g}.$$

Furthermore, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ into eigenspaces of s must satisfy $\mathfrak{h} = [\mathfrak{p}, \mathfrak{p}]$.

As described above, any symmetric space defines a symmetric triple. On the other hand, given a symmetric triple (\mathfrak{g}, s, B) , one can construct a globally symmetric space in the following way. Let G be a simply connected Lie group with Lie algebra \mathfrak{g} , and let $S \in \text{Aut}(G)$ be the automorphism with derivative s . The space $H = G^S \subset G$ of fixed points of S is closed and connected, and therefore the quotient manifold $M = G/H$ is simply connected by the long exact homotopy sequence. The form $B_{\mathfrak{p}}$ is invariant under the adjoint action of H on \mathfrak{p} , and so it defines a G -invariant metric on M . This gives M the structure of a symmetric space, and the associated symmetric triple is isomorphic to the original (\mathfrak{g}, s, B) . In fact, the constructions realize the following correspondence [CP].

Theorem 4.4. *There is a bijective correspondence between symmetric triples and simply connected globally symmetric spaces.*

Given a weight system defined by a representation

$$\rho: \mathfrak{g} \rightarrow \text{End}(V)$$

of a metrized real Lie algebra \mathfrak{g} , the constructions above tell us how the weight tensor $\rho(C) \in \text{End}(V) \otimes \text{End}(V)$ might be realized on a symmetric space. The point is that the weight tensor knows about the holonomy Lie algebra. There must exist some non-degenerate bilinear form B_V on V such that by lowering the indices of $\rho(C)$, we obtain a tensor $\check{R}^\rho \in (V^*)^{\otimes 4}$ with all the symmetries of a curvature tensor. First of all, it must satisfy

$$\check{R}_{cdab}^\rho = \check{R}_{abcd}^\rho,$$

which is automatic from the symmetry of $C \in \mathfrak{g} \otimes \mathfrak{g}$. Secondly, it must satisfy

$$\check{R}_{abcd}^\rho = -\check{R}_{bacd}^\rho, \tag{21}$$

which holds, for instance, if the representation ρ is skew-adjoint with respect to B_V . Finally, the tensor \check{R}^ρ must satisfy the algebraic Bianchi identity,

$$\check{R}_{abcd}^\rho + \check{R}_{bcad}^\rho + \check{R}_{cabd}^\rho = 0. \tag{22}$$

For convenience, we record the properties required of ρ in the following definition.

Definition 4.5. A representation $\rho: \mathfrak{g} \rightarrow \text{End}(V)$ of a metrized Lie algebra \mathfrak{g} , with Casimir tensor $C \in \mathfrak{g} \otimes \mathfrak{g}$, on a vector space V with a non-degenerate bilinear form B , is said to have *curvature symmetries* if the tensor $\check{R}^\rho \in (V^*)^{\otimes 4}$ given by

$$\check{R}_{abcd}^\rho = \rho(C)_{ac}^{xy} B_{xb} B_{yd}$$

satisfies the skew-symmetry in (21) and the Bianchi identity in (22).

These necessary symmetries are in fact sufficient to realize a Lie algebra weight tensor $\rho(C)$ on a symmetric space. Indeed, we can use the tensors B_V and \check{R}^ρ , as well as the corresponding $R^\rho \in (V^* \wedge V^*) \otimes \text{End}(V)$, with a raised index, to construct a Lie algebra $\mathfrak{h} = R^\rho(V \wedge V)$ and a symmetric triple $(\mathfrak{h} \oplus V, s, B)$, exactly as we did using the metric

and curvature on a single tangent space of a symmetric space. To prove that the bracket on $\mathfrak{h} \oplus V$ satisfies the Jacobi identity, both the Bianchi identity and the 4T relation are needed, but the latter is automatically satisfied for the tensor $\rho(C)$, as noted in (13). The Lie algebra \mathfrak{h} acts on (V, B_V) by skew-symmetric endomorphisms, and the weight tensor of this representation reproduces $\rho(C)$. In other words, the weight tensor must be realizable through an orthogonal representation and this representation must satisfy the Bianchi identity in the sense of Definition 4.5.

By applying Theorem 4.4, we have proved the following.

Theorem 4.6. *The weight tensor of a real Lie algebra weight system can be realized on a symmetric space if and only if it comes from a representation with curvature symmetries.*

Example 4.7. The weight tensor obtained from the standard representation of \mathfrak{sl}_2 (Example 2.7) cannot be realized on a symmetric space. Indeed, by the discussions above, the corresponding holonomy algebra would be a subalgebra of the one-dimensional Lie algebra \mathfrak{so}_2 , which certainly does not realize the weight tensor (15). \diamond

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